3. Higher-Order Linear Ordinary Differential Equations

3.1 Standard Form and Solution Space

Standard Form: An n-th order linear differential equation is represented in standard form as:

> **ODE:** $y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \dots + p_1(x)\, y' + p_0(x)\, y = r(x)$

- **Homogeneous ODE:** If $r(x) \equiv 0$, the solutions form an n-dimensional vector space.
- **Linear Independence of n Functions:** The set of functions $\{y_1, \dots, y_n\}$ is linearly independent on an interval I iff:

> **Equation:** $k_1 y_1(x) + k_2 y_2(x) + \dots + k_n y_n(x) = 0$ for all $x \in I \Rightarrow k_1 = k_2 = \dots = k_n = 0$

3.2 Generalized Wronskian

Generalized Wronskian: To verify linear independence of n solutions of the homogeneous equation, the generalized Wronskian determinant is defined as:

> $W(y_1, y_2, \dots, y_n)(x) = \det([y_k^{(j-1)}])$
> $W(y_1, \dots, y_n) = \det([[y_1, y_2, \dots, y_n], [y_1', y_2', \dots, y_n'], \dots, [y_1^{(n-1)}, y_2^{(n-1)}, \dots, y_n^{(n-1)}]])$

- **Wronskian Test for Independence Theorem:** Let $y_1, \dots, y_n$ be solutions of the homogeneous equation. They form a fundamental set (basis) iff $W \neq 0$ at some point x_0 in the interval I.

3.3 Constant Coefficient Homogeneous ODEs

Characteristic Equation: For $y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$, the characteristic equation is:

> **Equation:** $\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1 \lambda + a_0 = 0$

- **Distinct Real Roots:** For n distinct real roots $\lambda_1, \dots, \lambda_n$, the basis is $\{e^{\lambda_1 x}, \dots, e^{\lambda_n x}\}$.
- **Multiple Real Roots:** If a real root λ has multiplicity m, the corresponding basis solutions are:

> **Basis Solutions:** $e^{\lambda x},\ x e^{\lambda x},\ x^2 e^{\lambda x},\ \dots,\ x^{m-1} e^{\lambda x}$

- **Complex Multiplicity:** If a complex conjugate pair $\alpha \pm i\beta$ has multiplicity m, the basis includes:

> **Basis Solutions:** $x^j e^{\alpha x} \cos(\beta x),\ x^j e^{\alpha x} \sin(\beta x)$ (for $j = 0, 1, \dots, m-1$)

3.4 Variation of Parameters for Third-Order ODEs

Method Formulation: For $y''' + p(x)y'' + q(x)y' + r(x)y = f(x)$, with known homogeneous basis $\{y_1, y_2, y_3\}$:

> **Particular Solution:** $y_p = u_1(x)\, y_1 + u_2(x)\, y_2 + u_3(x)\, y_3$
> **System Eq 1:** $u_1' y_1 + u_2' y_2 + u_3' y_3 = 0$
> **System Eq 2:** $u_1' y_1' + u_2' y_2' + u_3' y_3' = 0$
> **System Eq 3:** $u_1' y_1'' + u_2' y_2'' + u_3' y_3'' = f(x)$

Solved using Cramer's rule: $u_j'(x) = W_j(x) / W(x)$. Integrating u_j' gives the coefficients.

3.5 Cauchy-Euler (Equidimensional) Equations

Standard Form: Equations of the form: $ax^2 y'' + bx y' + cy = 0$ (where $a \neq 0$, $x > 0$).

- **Trial Solution:** $y = x^m$, which yields the indicial equation:

> **Indicial Equation:** $a\, m(m - 1) + b\, m + c = 0 \Rightarrow a\, m^2 + (b - a)\, m + c = 0$

- **Case 1:** Distinct Real Roots ($m_1 \neq m_2$):

> **General Solution:** $y = c_1\, x^{m_1} + c_2\, x^{m_2}$

- **Case 2:** Repeated Real Root ($m_1 = m_2 = m$):

> **General Solution:** $y = (c_1 + c_2 \ln x)\, x^m$

- **Case 3:** Complex Conjugate Roots ($m = \alpha \pm i\beta$):

> **General Solution:** $y = x^\alpha [\, c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)\,]$